EIGENVALUE PINCHING AND APPLICATION TO THE STABILITY AND THE ALMOST UMBILICITY OF HYPERSURFACES

J.-F. GROSJEAN

Abstract. In this paper we give pinching theorems for the first nonzero eigenvalue of the Laplacian on the compact hypersurfaces of ambient spaces with bounded sectional curvature. As application we deduce a rigidity result for stable constant mean curvature hypersurfaces M of these spaces N. Indeed, we prove that if M is included in a ball of radius small enough then the Hausdorff-distance between M and a geodesic sphere S of N is small. Moreover M is diffeomorphic and quasi-isometric to S. As other application, we obtain rigidity results for almost umbilic hypersurfaces.

Contents

1. Introduction 1
2. Preliminaries 5
3. An L^2-approach 6
4. The proof of the diffeomorphism 14
5. Application to the stability 20
6. Application to the almost umbilic hypersurfaces 22

References 23

1. Introduction

One way to show that the geodesic spheres are the only stable constant mean curvature hypersurfaces of classical model spaces (i.e. the Euclidean space, the spherical space and the hyperbolic space) is to prove that there is equality in the well-known Reilly's inequality. One of the main points of the present paper is to obtain new stability results for hypersufaces immersed in more general ambient spaces.

First, let us recall the Reilly's inequality. Let (M^m, g) be a compact, connected and oriented m-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in the simply connected model space $N^{n+1}(c)$ ($c = 0$, 1 ,-1 respectively for the Euclidean space, the sphere or the hyperbolic space). The Reilly's inequality gives an extrinsic upper bound for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian of (M^m, g) in term of the square of the length of the mean curvature H. Indeed we have

Date: 10th October 2008.
2000 *Mathematics Subject Classification.* 53A07, 53C21.
Key words and phrases. Spectrum, Laplacian, pinching results, hypersurfaces.

$$\lambda_1(M) \leqslant \frac{m}{V(M)} \int_M (|H|^2 + c) dv \tag{1}$$

where dv and $V(M)$ denote respectively the Riemannian volume element and the volume of (M^m, g). Moreover in the case of hypersurfaces (i.e. $m = n$), the equality holds if and only if (M^n, g) is immersed as a geodesic sphere of $N^{n+1}(c)$. For $c = 0$ this inequality was proved by Reilly ([11]) and can easily be extended to the spherical case $c = 1$ by considering the canonical embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. For $c = -1$ it has been proved by El Soufi and Ilias in [7].

In the sequel we will consider a weaker inequality due to Heintze ([8]) which generalizes the previous for the case where (M^m, g) is isometrically immersed by ϕ in a $n+1$-dimensional Riemannian manifold (N^{n+1}, h) whose sectional curvature K^N is bounded above by δ. Indeed if $\phi(M)$ lies in a convex ball and if the radius of this ball is $\frac{\pi}{4\sqrt{\delta}}$ in the case $\delta > 0$, we have

$$\lambda_1(M) \leqslant m(\|H\|_\infty^2 + \delta) \tag{2}$$

where $\|H\|_\infty$ denotes the L^∞-norm of the mean curvature. Now for $m = n$ if we assume that K^N is bounded below by μ and M has a constant mean curvature H and is stable (see section 5) we have

$$n(H^2 + \mu) \leqslant \lambda_1(M) \leqslant n(H^2 + \delta)$$

Consequently we see that if N is not of constant sectional curvature we can't conclude as in the model spaces. However, the above inequality is a kind of pinching on the Reilly's inequality, that is a condition of almost equality. Such conditions have been studied for the Reilly's inequality in the Euclidean space in [6]. In the present paper we will generalize the results of [6] to the inequality (2) for hypersurfaces (i.e. $m = n$) of ambient spaces with non constant sectional curvature. That amounts to finding a constant C depending on minimum geometric invariants so that if we have the condition

$$(P_C) \qquad n(\|H\|_\infty^2 + \delta)(1 - C) < \lambda_1(M)$$

then M is close to a sphere in a certain sense.

Before giving the main theorems, we precise some notations which will be more convenient. Throughout the paper, we will note $h = (\|H\|_\infty^2 + \delta)^{1/2}$ and B the second fundamental form. Moreover if (N^{n+1}, h) is a $n+1$-dimensional Riemannian manifold so that $K^N \leqslant \delta$ and the injectivity radius $i(N)$ satisfies $i(N) \geqslant \frac{\pi}{\sqrt{\delta}}$ if $\delta > 0$, we will note $\mathcal{H}^\star(n, N)$ the space of all Riemannian compact, connected and oriented n-dimensional Riemannian manifolds without boundary isometrically immersed by ϕ in (N^{n+1}, h). We call $\mathcal{H}_C(n, N)$ the space of all Riemannian manifolds of $\mathcal{H}^\star(n, N)$ satisfying the following convexity hypothesis : $\phi(M)$ lies in a convex ball and the radius of this ball is $\frac{\pi}{4\sqrt{\delta}}$ if $\delta > 0$. Moreover $\mathcal{H}_V(n, N)$ will be the space of all Riemannian manifolds of $\mathcal{H}^\star(n, N)$ which satisfy the following hypothesis on the volume : $V(M) \leqslant \frac{c\omega_n}{\delta^{n/2}}$ if $\delta > 0$ and $V(M) \leqslant c\omega_n i(N)^n$ if $\delta \leqslant 0$ for some constant c. These two hypotheses on the volume of M with the condition on $i(N)$ for $\delta > 0$

are providing from hypotheses assumed in a result on a Sobolev inequality due to Hoffman and Spruck ([9] and [10]). At last we put $\mathcal{H}(n,N) = \mathcal{H}_C(n,N) \cap \mathcal{H}_V(n,N)$.

Furthermore we need the following function s_δ defined by

$$s_\delta(r) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin \sqrt{\delta} r & \text{if } \delta > 0 \\ r & \text{if } \delta = 0 \\ \frac{1}{\sqrt{|\delta|}} \sinh \sqrt{|\delta|} r & \text{if } \delta < 0 \end{cases},$$

Moreover we will note $B(p,R)$ all geodesic ball in N of center p and radius R.

Let us state the first main theorem.

Theorem 1.1. *Let (N^{n+1}, h) be a $n+1$-dimensional Riemannian manifold whose sectional curvature K^N satisfies $\mu \leqslant K^N \leqslant \delta$ and $i(N) \geqslant \frac{\pi}{\sqrt{\delta}}$ if $\delta > 0$ and let $M \in \mathcal{H}(n,N)$. Let $\varepsilon \leqslant 1/18$. Then there exist a point p and positive constants $C_\varepsilon(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta, \mu)$ and $R(\delta,\mu,\varepsilon)$ so that if $\phi(M)$ is contained in the ball $B(p, R(\delta,\mu,\varepsilon))$ and if the pinching condition (P_{C_ε})*

$$n(\|H\|_\infty^2 + \delta)(1 - C_\varepsilon) < \lambda_1(M)$$

is satisfied then M is ε-Gromov-Hausdorff close to $S(p, s_\delta^{-1}\left(\frac{1}{h}\right))$. Namely the Gromov-Hausdorff distance satisfies

$$d_{GH}\left(\phi(M), S\left(p, s_\delta^{-1}\left(\frac{1}{h}\right)\right)\right) \leqslant \frac{\varepsilon}{h}$$

and M is diffeomorphic and ε-quasi-isometric to $S(p, s_\delta^{-1}\left(\frac{1}{h}\right))$. Namely there exists a diffeomorphism from M into $S(p, s_\delta^{-1}\left(\frac{1}{h}\right))$ so that

$$\left| |dF_x(u)|^2 - 1 \right| \leqslant \varepsilon$$

for any $x \in M$, $u \in T_x M$ and $|u| = 1$.

Moreover, $R(\delta,\mu,\varepsilon) \longrightarrow \infty$ when $\delta - \mu \longrightarrow 0$. On the other hand,

1. *$C_\varepsilon(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta) \longrightarrow 0$ when $\varepsilon \longrightarrow 0$.*
2. *$C_\varepsilon(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta) \longrightarrow 0$ when $\|H\|_\infty \longrightarrow \infty$.*
3. *if $V(M)^{1/n}\|B\|_\infty \leqslant v$, then $h^2 C_\varepsilon(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta) \longrightarrow \infty$ when $\|H\|_\infty \longrightarrow \infty$.*

We recall that the Gromov-Hausdorff distance between two compact subsets A and B of a metric space is given by

$$d_{GH}(A,B) = \inf\{A \subset V_\eta(B) \text{ and } B \subset V_\eta(A)\}$$

where for any subset A, $V_\eta(A)$ is the tubular neighborhood of A defined by $V_\eta(A) = \{x | d(x,A) < \eta\}$.

Remark 1.1. *The point p is not depending on ε, $\|H\|_\infty$ or $\|B\|_\infty$. The point p is nothing but the center of mass of M (see preliminaries).*

On the other hand, for $\delta \geqslant 0$, we can omitted the dependence on $\|H\|_\infty$.

As in the euclidean case (see [6]), in the hyperbolic case or spherical case, we can obtain the Hausdorff proximity strictly with a dependence on $\|H\|_\infty$. More precisely we have the

Theorem 1.2. *Let $N^{n+1}(\delta)$ with $\delta = -1, 0$ or 1 where $N^{n+1}(-1)$, $N^{n+1}(0)$ and $N^{n+1}(1)$ are respectively the hyperbolic space, the euclidean space and the sphere. Let $M \in \mathcal{H}(n, N)$. Then for any $\varepsilon > 0$ there exists a positive constant $C_\varepsilon(n, \|H\|_\infty, V(M))$ so that if the pinching condition (P_{C_ε})*

$$n(\|H\|_\infty^2 + \delta)(1 - C_\varepsilon) < \lambda_1(M)$$

then M is ε-Gromov-Hausdorff close to $S(p, s_\delta^{-1}\left(\frac{1}{h}\right))$.

The condition (3) of the theorem 1.1 allows to obtain an application for the stable constant mean curvature hypersurfaces. Indeed we have the following stability theorem

Theorem 1.3. *Let (N^{n+1}, h) be a $n+1$-dimensional Riemannian manifold whose sectional curvature K^N satisfies $\mu \leqslant K^N \leqslant \delta$ and $i(N) \geqslant \frac{\pi}{\sqrt{\delta}}$ if $\delta > 0$ and let $M \in \mathcal{H}_C(n, N)$. Let $v > 0$ so that $V(M)^{1/n}\|B\|_\infty \leqslant v$. For any $\varepsilon < 1/18$, there exists a constant $R_\varepsilon(\delta, \mu, v, i(N)) > 0$ so that if $\phi(M)$ lies in a ball of radius $R_\varepsilon(\delta, \mu, v, i(N))$ and ϕ is of constant mean curvature H and is stable then there exists a point p so that M is ε-Gromov-Hausdorff close, diffeomorphic and ε-quasi-isometric to $S(p, s_\delta^{-1}\left(\frac{1}{h}\right))$.*

Remark 1.2. *If $\delta > 0$, $R_\varepsilon(\delta, \mu, v, i(N))$ is not depending on $i(N)$.*

As another application of theorems 1.1 and 1.2 we have results for the almost umbilic hypersurfaces of model spaces. These theorems are to compare with results of Shiohama and Xu ([14] and [15]) which obtain conditions on the Betti numbers.

Theorem 1.4. *Let (N^{n+1}, h) be a $n+1$-dimensional Riemannian manifold with constant sectional curvature δ and let $M \in \mathcal{H}(n, N)$. Let p be the center of mass of M. Let $q > \frac{n}{2}$ and $\varepsilon < 1/18$. Then there exist positive constants $\eta_{1,\varepsilon}(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta)$ and $\eta_{2,\varepsilon}(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta)$ so that if*

1. $\|\tau\|_{2q} \leqslant \eta_{1,\varepsilon}\|H\|_\infty V(M)^{1/2q}$.
2. $\left\|H^2 - \|H\|_\infty^2\right\|_q \leqslant \eta_{2,\varepsilon}\|H\|_\infty^2 V(M)^{1/q}$.

Then M is ε-Gromov-Hausdorff close, diffeomorphic and ε-quasi-isometric to $S(p, s_\delta^{-1}\left(\frac{1}{h}\right))$.

Remark 1.3. *The dependence on $\|B\|_\infty$ is not necessary for the Hausdorff proximity.*

In the Euclidean case providing from the pinching theorem proved in [6] we can improve the condition 2)

Theorem 1.5. *Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Let p be the center of mass of M. Then for any $\varepsilon > 0$, there exist two constants $\eta_{1,\varepsilon}(n, \|H\|_\infty, V(M))$ and $\eta_{2,\varepsilon}(n, \|H\|_\infty, V(M))$ so that if*

1. $\|\tau\|_{2q} \leqslant \eta_{1,\varepsilon}\|H\|_\infty V(M)^{1/2q}$.
2. $\left\|H^2 - \frac{\|H\|_{2r}^2}{V(M)^{1/r}}\right\|_q \leqslant \eta_{2,\varepsilon}\|H\|_\infty^2 V(M)^{1/q}$ *for $r \geqslant 2$.*

Then M is ε-Gromov-Hausdorff close to $S\left(p, \frac{V(M)^{1/2r}}{\|H\|_{2r}}\right)$. Moreover there exist two constants $\eta_{1,\varepsilon}(n, \|B\|_\infty, V(M))$ and $\eta_{2,\varepsilon}(n, \|B\|_\infty, V(M))$ so that if the two conditions 1) and 2) (by replacing $\|H\|_\infty$ by $\|B\|_\infty$) are satisfied then M is diffeomorphic and ε-quasi-isometric to $S\left(p, \frac{V(M)^{1/2r}}{\|H\|_{2r}}\right)$.

2. Preliminaries

Let (M^n, g) be a compact, connected n-dimensional Riemannian manifold isometrically immersed by ϕ in an $n+1$-dimensional Riemannian manifold (N^{n+1}, h) which sectional curvature is bounded by δ. For any point $p \in N$ let us consider exp be the exponential map at this point. Locally we consider $(x_i)_{1\leqslant i\leqslant n}$ the normal coordinates of N centered at p and for all $x \in N$, we denote by $r(x) = d(p, x)$, the geodesic distance between p and x on (N^{n+1}, h).

Moreover we define the function c_δ by $c_\delta = s'_\delta$. Obviously, we have $c_\delta^2 + \delta s_\delta^2 = 1$ and $c'_\delta = -\delta s_\delta$.

The gradient of a function u define on N with respect to h will be denoted by $\nabla^N u$ and the gradient with respect to g of the restriction of u on M will be denoted by $\nabla^M u$.

Briefly, we recall the proof of Heintze ([8]) for the Reilly's inequality.

We will use $\frac{s_\delta(r)}{r} x_i$ as test functions in the variational characterization of $\lambda_1(M)$. But these functions must be L^2-orthogonal to the constant functions. For this purpose, we use a standard argument used by Chavel and Heintze ([5] and [8]). Indeed, if $\phi(M)$ lies in a convex ball we can define the vector field Y by

$$Y_q = \int_M \frac{s_\delta(d(q,x))}{d(q,x)} exp_q^{-1}(x) dv(x) \in T_q N, \quad q \in M \quad ,$$

From the fixed point theorem of Brouwer, there exists a point $p \in N$ such that $Y_p = 0$ and consequently, for a such p, $\displaystyle\int_M \frac{s_\delta(r)}{r} x_i dv = 0$. For $\delta > 0$, we assume in addition that $\phi(M)$ is contained in a ball of radius $\frac{\pi}{4\sqrt{\delta}}$. Indeed, in this case $\phi(M)$ lies in a ball of center p (the point p so that $Y_p = 0$) with a radius less or equal to $\frac{\pi}{2\sqrt{\delta}}$ and c_δ is then a nonnegative function.

Now considering the vector field on M, $Z = s_\delta \nabla^N r$ and noting that the coordinates of Z in the normal local frame are $\left(\frac{s_\delta(r)}{r} x_i\right)_{1\leqslant i\leqslant n}$, we have

$$\begin{aligned}\lambda_1(M)\int_M s_\delta^2(r) dv = \lambda_1(M)\int_M |Z|^2 dv = \lambda_1(M)\int_M \sum_{i=1}^{n+1}\left(\frac{s_\delta(r)}{r}x_i\right)^2 dv \\ \leqslant \int_M \sum_{i=1}^{n+1}\left|\nabla^M\left(\frac{s_\delta(r)}{r}x_i\right)\right|^2 dv\end{aligned}$$

Now, Heintze proved that $\displaystyle\sum_{i=1}^{n+1}\left|\nabla^M\left(\frac{s_\delta(r)}{r}x_i\right)\right|^2 \leqslant n - \delta|Z^T|^2$ and

$$\text{div }(Z^T) \geqslant nc_\delta - nH\langle Z, \nu\rangle \tag{3}$$

Then

$$\begin{aligned}\lambda_1(M)\int_M |Z|^2 dv &\leqslant \int_M (n-\delta|Z^T|^2)dv\\ &= \int_M (n-\text{div }(Z^T)c_\delta)dv\\ &\leqslant \int_M (n-nc_\delta^2+nH\langle Z,\nu\rangle c_\delta)dv\\ &= \int_M (n\delta s_\delta^2+nH\langle Z,\nu\rangle c_\delta)dv\\ &\leqslant \int_M n\delta s_\delta^2 dv+\|H\|_\infty\int_M n s_\delta c_\delta dv\end{aligned}$$

and using again (3) we get

$$\begin{aligned}\lambda_1(M)\int_M |Z|^2 dv &\leqslant n\delta\int_M |Z|^2dv+\|H\|_\infty\int_M (nH\langle Z,\nu\rangle s_\delta+\text{div }(Z^T)s_\delta)dv\\ &= n\delta\int_M |Z|^2dv+\|H\|_\infty\int_M (nH\langle Z,\nu\rangle s_\delta-c_\delta s_\delta|\nabla^M r|^2)dv\\ &\leqslant n\delta\int_M |Z|^2dv+n\|H\|_\infty^2\int_M |\langle Z,\nu\rangle||Z|dv\\ &\leqslant n\delta\int_M |Z|^2dv+n\|H\|_\infty^2\int_M |Z|^2dv\\ &\leqslant n(\|H\|_\infty^2+\delta)\int_M |Z|^2dv\end{aligned}$$

3. An L^2-approach

Throughout the paper we assume that $\phi(M)$ is included in a ball of radius $\frac{\pi}{4\sqrt{\delta}}$ for $\delta>0$. Note that we have

$$\frac{|\delta|}{\|H\|_\infty^2}\leqslant 1 \tag{4}$$

This inequality is obvious for $\delta\leqslant 0$. For $\delta>0$, as we have assumed that $\phi(M)$ is in a ball of radius $\frac{\pi}{4\sqrt{\delta}}$, it follows that $\frac{s_\delta\left(\frac{\pi}{4\sqrt{\delta}}\right)}{c_\delta\left(\frac{\pi}{4\sqrt{\delta}}\right)}\geqslant\frac{1}{\|H\|_\infty}$ and then $\frac{\delta}{\|H\|_\infty^2}\leqslant 1$.

Moreover α will denote a constant depending only on n.

Lemma 3.1. *If the pinching condition (P_C) holds then $\|Z^T\|_2^2\leqslant\frac{h^2}{\|H\|_\infty^2}\|Z\|_2^2C$.*

Proof. We have

$$\|Z^T\|_2^2=\int_M |Z|^2dv-\int_M\langle Z,\nu\rangle^2dv\leqslant\int_M(|Z|^2-|\langle Z,\nu\rangle||Z|)dv$$

and from the proof of Reilly's inequality and the pinching condition, we have

$$n\|H\|_\infty^2\left(\int_M |Z|^2dv - \int_M |\langle Z,\nu\rangle||Z|dv\right) < nh^2C\|Z\|_2^2$$

□

Lemma 3.2. *If $C < \frac{n}{2(n+1)}$, then (P_C) implies that*

$$\|Z\|_2^2 \leqslant \left(\frac{n}{n-(n+1)C}\right)\frac{V(M)}{h^2} \leqslant \frac{2V(M)}{h^2}$$

Proof. From the proof of the Reilly's inequality we have

$$\begin{aligned}\lambda_1(M)\|Z\|_2^2 &\leqslant nV(M) - \delta\|Z^T\|_2^2\\ &\leqslant nV(M) + \frac{|\delta|}{\|H\|_\infty^2}h^2C\|Z\|_2^2\end{aligned}$$

From (4) and the pinching condition we have

$$h^2(n-(n+1)C)\|Z\|_2^2 \leqslant nV(M)$$

and the condition on C allows us to obtain the desired inequality.

□

Lemma 3.3. *If $C < \frac{n}{2(n+1)}$ and if the pinching condition (P_C) holds then $\|Z\|_2^2 \geqslant \frac{V(M)}{4h^2}$.*

Proof. From the proof of the Reilly's inequality, we have

$$\begin{aligned}\lambda_1(M)\|Z\|_2^2 &\leqslant nV(M) - \delta\|Z^T\|_2^2\\ &\leqslant \frac{(nV(M)-\delta\|Z^T\|_2^2)^2}{nV(M)-\delta\|Z^T\|_2^2}\\ &\leqslant \frac{n^2h^4\|Z\|_2^4}{nV(M)-\frac{|\delta|h^2}{\|H\|_\infty^2}\|Z\|_2^2C}\end{aligned}$$

and using successively (4), the pinching condition and the previous lemma we get

$$\begin{aligned}\|Z\|_2^2 &\geqslant \frac{1}{h^2}(1-C)\left(1-\frac{2}{n}C\right)V(M)\\ &\geqslant \frac{(1-C)^2}{h^2}V(M) \geqslant \frac{V(M)}{4h^2}\end{aligned}$$

□

Lemma 3.4. *Let $X = nHc_\delta\nu - n\|H\|_\infty^2 Z$. If $C < \frac{n}{2(n+1)}$, then the pinching condition (P_C) implies $\|X\|_2^2 \leqslant \alpha\|H\|_\infty^2 V(M)C$.*

Proof. Using again (3) and the previous lemmas we have

$$\|X\|_2^2 = n^2\int_M H^2c_\delta^2dv - 2n^2\|H\|_\infty^2\int_M H\langle Z,\nu\rangle c_\delta dv + n^2\|H\|_\infty^4\|Z\|_2^2$$

$$\begin{aligned}
&\leqslant n^2\int_M H^2c_\delta^2dv + 2n\|H\|_\infty^2\int_M(\mathrm{div}\,(Z^T)c_\delta - nc_\delta^2)dv + n^2\|H\|_\infty^4\|Z\|_2^2\\
&= n^2\int_M H^2c_\delta^2dv + 2n\|H\|_\infty^2\int_M \delta|Z^T|^2dv - 2n^2\|H\|_\infty^2\int_M c_\delta^2dv + n^2\|H\|_\infty^4\|Z\|_2^2\\
&\leqslant -n^2\|H\|_\infty^2\int_M c_\delta^2dv + n^2\|H\|_\infty^4\|Z\|_2^2 + 2n|\delta|\|H\|_\infty^2\|Z^T\|_2^2\\
&= -n^2\|H\|_\infty^2V(M) + n^2\|H\|_\infty^2\delta\|Z\|_2^2 + n^2\|H\|_\infty^4\|Z\|_2^2 + 2n|\delta|h^2\|Z\|_2^2C\\
&= n^2\|H\|_\infty^2(-V(M) + h^2\|Z\|_2^2) + 4n|\delta|V(M)C\\
&\leqslant n^2\|H\|_\infty^2\left(-1+\frac{n}{n-(n+1)C}\right)V(M) + 4n|\delta|V(M)C\\
&= \|H\|_\infty^2\left(n^2(n+1) + 4n\frac{|\delta|}{\|H\|_\infty^2}\right)V(M)C
\end{aligned}$$

And we complete the proof by applying again (4). $\square$

Lemma 3.5. *Let* $Y = |Z|^{1/2}\left(\delta Z + Hc_\delta\nu - h\frac{Z}{|Z|}\right)$. *If* $C < \frac{n}{2(n+1)}$, *then the condition* (P_C) *implies*

$$\|Y\|_2^2 \leqslant \alpha\left(h + \frac{\|H\|_\infty^2}{h}\max(1,\|H\|_\infty^\gamma V(M)^{\gamma/n})\right)V(M)C$$

where $\gamma \in (e^{n/2}-1, e^n-1)$.

Proof. First we have

$$\begin{aligned}
\|Y\|_2^2 &\leqslant \int_M\left(|Z||\delta Z + Hc_\delta\nu|^2 - 2h\langle\delta Z + Hc_\delta\nu, Z\rangle + h^2|Z|\right)dv\\
&\leqslant \int_M\left(|Z||\delta Z + Hc_\delta\nu|^2 - 2h\langle\delta Z + Hc_\delta\nu, Z\rangle\right)dv + h^2\|Z\|_2V(M)^{1/2} \quad (5)
\end{aligned}$$

Let us compute the first term

$$\begin{aligned}
\int_M |Z||\delta Z + Hc_\delta\nu|^2dv &= \int_M |Z|\left(\delta^2|Z|^2 + 2\delta c_\delta H\langle Z,\nu\rangle + H^2c_\delta^2\right)dv\\
&= \int_M |Z|\left(H^2 - \delta H^2s_\delta^2 + 2\delta c_\delta H\langle Z,\nu\rangle + \delta - \delta c_\delta^2\right)dv\\
&= \int_M |Z|(H^2+\delta-\delta|HZ - c_\delta\nu|^2)dv\\
&\leqslant h^2\|Z\|_2V(M)^{1/2} - \delta\int_M|Z||HZ - c_\delta\nu|^2dv\\
&\leqslant h^2\|Z\|_2V(M)^{1/2} + |\delta|\|Z\|_\infty\int_M|HZ - c_\delta\nu|^2dv
\end{aligned}$$

On the other hand,

$$\int_M |HZ - c_\delta \nu|^2 dv \leqslant \|H\|_\infty^2 \int_M s_\delta^2 dv - 2\int_M H\langle Z,\nu\rangle c_\delta dv + \int_M c_\delta^2 dv$$

Now the pinching implies that

$$\|H\|_\infty^2 \int_M s_\delta^2 dv - \int_M H\langle Z,\nu\rangle c_\delta dv \leqslant nh^2\|Z\|_2^2 C \leqslant 2nV(M)C$$

and

$$\begin{aligned}\int_M c_\delta^2 dv - \int_M H\langle Z,\nu\rangle c_\delta dv &\leqslant \frac{1}{n}\int_M \text{div }(Z^T)c_\delta dv \leqslant \frac{|\delta|}{n}\|Z^T\|_2^2 \\ &\leqslant \frac{2|\delta|}{n\|H\|_\infty^2}V(M)C \leqslant \frac{2}{n}V(M)C\end{aligned}$$

Then we have proved

$$\text{(6)} \qquad \int_M |Z||\delta Z + Hc_\delta \nu|^2 dv \leqslant h^2\|Z\|_2 V(M)^{1/2} + \alpha|\delta|\|Z\|_\infty V(M)C$$

Now let us compute the two last terms of (5)

$$\begin{aligned}&-2h\int_M \langle \delta Z + Hc_\delta\nu, Z\rangle dv + h^2\|Z\|_2 V(M)^{1/2} \\ &\leqslant -2\delta h\int_M s_\delta^2 dv + \frac{2h}{n}\int_M \text{div }(Z^T)c_\delta dv - 2h\int_M c_\delta^2 dv + h^2\|Z\|_2 V(M)^{1/2} \\ &= -2hV(M) + \frac{2h\delta}{n}\int_M |Z^T|^2 dv + h^2\|Z\|_2 V(M)^{1/2} \\ &\leqslant -2hV(M) + h^2 V(M)^{1/2}\end{aligned}$$

Therefore reporting this and (6) in (5), we get

$$\|Y\|_2^2 \leqslant 2h^2\|Z\|_2 V(M)^{1/2} - 2hV(M) + \frac{2h\delta}{n}\|Z^T\|_2^2 + \alpha|\delta|\|Z\|_\infty V(M)C$$

and using the previous lemmas, we get

$$\begin{aligned}\|Y\|_2^2 &\leqslant 2h\left(\left(\frac{n}{n-(n+1)C}\right)^{1/2} - 1\right)V(M) + \left(\alpha|\delta|\|Z\|_\infty + \frac{4h|\delta|}{n\|H\|_\infty^2}\right)V(M)C \\ &\leqslant \left(4h + \alpha|\delta|\|Z\|_\infty + \frac{4h|\delta|}{n\|H\|_\infty^2}\right)V(M)C \\ &\leqslant \alpha(h + |\delta|\|Z\|_\infty)V(M)C\end{aligned}$$

Now the researched inequality is a straightforward consequence of the following lemma

Lemma 3.6. *If $C < \frac{n}{2(n+1)}$, then the pinching condition (P_C) implies*

$$\|Z\|_\infty \leqslant \frac{\alpha}{h}\overline{Z}(n, \|H\|_\infty, V(M))$$

where $\overline{Z} = \max(1, \|H\|_\infty^\gamma V(M)^{\gamma/n})$.

Then

$$\begin{aligned}\|Y\|_2^2 &\leqslant \alpha\left(h + \frac{|\delta|}{h}\max(1, \|H\|_\infty^\gamma V(M)^{\gamma/n})\right)V(M)C\\ &\leqslant \alpha\left(h + \frac{\|H\|_\infty^2}{h}\max(1, \|H\|_\infty^\gamma V(M)^{\gamma/n}\right)V(M)C\end{aligned}$$

□

The proof of the lemma 3.6 is providing from a result stated in the following proposition using a Nirenberg-Moser type of proof (see [6]).

Proposition 3.1. *Let (N^{n+1}, h) be a Riemmannian manifold so that $K^N \leqslant \delta$ and $i(N) \geqslant \frac{\pi}{\sqrt{\delta}}$ if $\delta > 0$ and let $M \in \mathcal{H}_V(n, N)$. Let ξ be a nonnegative continuous function so that ξ^k is smooth for $k \geqslant 2$. Let $0 \leqslant r < s \leqslant 2$ so that*

$$\frac{1}{2}\int_M \Delta\xi^2\xi^{2k-2}dv \leqslant (A_1 + kA_2)\int_M \xi^{2k-r}dv + (B_1 + kB_2)\int_M \xi^{2k-s}dv$$

where A_1, A_2, B_1, B_2 are nonnegative constants. Then for any $\eta > 0$, if $\|\xi\|_\infty > \eta$ then

$$\|\xi\|_\infty \leqslant L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, V(M), \eta)\|\xi\|_2$$

where

$$\begin{aligned}&L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, V(M), \eta)\\ &= K(n)\left(\frac{4A_1^{1/2} + 4A_2^{1/2}}{\eta^{r/2}} + \frac{4B_1^{1/2} + 4B_2^{1/2}}{\eta^{s/2}} + \|H\|_\infty\right)^\gamma V(M)^{\frac{\gamma}{n} - \frac{1}{2}}\end{aligned}$$

and $\gamma \in (e^{n/2} - 1, e^n - 1)$.

Remark 3.1. *In particular we see that*

1. *If* $\|\xi\|_2 \leqslant \dfrac{\eta}{L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, V(M), \eta)}$, *then* $\|\xi\|_\infty \leqslant \eta$.
2. *If* $\|\xi\|_2 \leqslant A$, *then for any* $\eta > 0$,

$$\|\xi\|_\infty \leqslant \max(\eta, L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, V(M), \eta)A)$$

In [6] this proposition has been proved for hypersurfaces of the Euclidean space. The proof is similar for hypersurfaces of some ambient space with bounded sectional curvature. This proof uses a Sobolev inequality due to Hoffman and Spruck (see [9] and [10]) which is available under the conditions on the injectivity radius of N and the volume of M contained in the definition of $\mathcal{H}_V(n, N)$.

Proof of the lemma 3.6: First we compute the Laplacian of $|Z|^2$. An easy computation shows that $\Delta|Z|^2 = (-2c_\delta^2 + 2\delta s_\delta^2)|\nabla^M r|^2 + 2s_\delta c_\delta \Delta r$.

Since $r \leqslant \frac{\pi}{4\sqrt{\delta}}$ for $\delta > 0$, the first term is nonpositif. Now let us consider $(e_i)_{1\leqslant i\leqslant n+1}$ an orthonormal frame in a neighborhood of the point $p \in M$ where we compute the Laplacian and so that $e_{n+1} = \nu$. Then

$$\begin{aligned}
\Delta|Z|^2 &\leqslant 2s_\delta c_\delta \left(-\sum_{i=1}^n \nabla^N\ dr(e_i, e_i) + nH\langle \nu, \nabla^N\ r\rangle \right) \\
&= 2s_\delta c_\delta \left(-\sum_{i=1}^n \nabla^N\ dr(e_i - \langle \nabla^N\ r, e_i\rangle \nabla^N\ r, e_i - \langle \nabla^N\ r, e_i\rangle \nabla^N\ r) + nH\langle \nu, \nabla^N\ r\rangle \right) \\
&\leqslant 2s_\delta c_\delta \left(-\frac{c_\delta}{s_\delta} \sum_{i=1}^n \left| e_i - \langle \nabla^N\ r, e_i\rangle \nabla^N\ r \right|^2 + n\|H\|_\infty \right) \\
&\leqslant 2n\|H\|_\infty s_\delta c_\delta \leqslant 2n\|H\|_\infty s_\delta (1 + \sqrt{|\delta|} s_\delta) \\
&= 2n\|H\|_\infty |Z| + 2n\|H\|_\infty \sqrt{|\delta|} |Z|^2
\end{aligned}$$

And from the remark 3.1 about the proposition 3.1 and lemma 3.2 we deduce that

$$\|Z\|_\infty \leqslant \max\left(\eta, L\left(n, 2n\|H\|_\infty, 0, 2n\|H\|_\infty \sqrt{|\delta|}, 0, \|H\|_\infty, \eta \right) \|Z\|_2 \right) \tag{7}$$

Now

$$\begin{aligned}
&L\left(n, 2n\|H\|_\infty, 0, 2n\|H\|_\infty \sqrt{|\delta|}, 0, \|H\|_\infty, \eta \right) \\
&= K(n) \left(\frac{4}{\eta^{1/2}} (2n\|H\|_\infty)^{1/2} + 4(2n\|H\|_\infty \sqrt{|\delta|})^{1/2} + \|H\|_\infty \right)^\gamma V(M)^{\frac{\gamma}{n} - \frac{1}{2}} \\
&= K(n) \left(\frac{4(2n)^{1/2}}{\eta^{1/2}\|H\|_\infty^{1/2}} + 4\frac{(2n\sqrt{|\delta|})^{1/2}}{\|H\|_\infty^{1/2}} + 1 \right)^\gamma \|H\|_\infty^\gamma V(M)^{\frac{\gamma}{n} - \frac{1}{2}}
\end{aligned}$$

and choosing $\eta = \frac{1}{h}$ with the fact that $\frac{|\delta|}{\|H\|_\infty^2} \leqslant 1$ we obtain that

$$L\left(n, 2n\|H\|_\infty, 0, 2n\|H\|_\infty \sqrt{|\delta|}, 0, \|H\|_\infty, \eta \right) \leqslant \alpha \|H\|_\infty^\gamma V(M)^{\frac{\gamma}{n} - \frac{1}{2}}$$

We conclude by reporting this in (7) and by using the lemma 3.2.

$\square$

Let's introduce now the function $\varphi = |Z| \left(|Z| - \frac{1}{h} \right)^2 = |Z| \left| Z - \frac{1}{h}\frac{Z}{|Z|} \right|^2$. In the following lemma, we give an L^2-estimate of φ

Lemma 3.7. *If $C < \frac{n}{2(n+1)}$, then (P_C) implies that*

$$\|\varphi\|_2 \leqslant \alpha \|\varphi\|_\infty^{3/4} K_1(n, \|H\|_\infty, V(M)) \|H\|_\infty^{1/2} \frac{V(M)^{1/2} C^{1/4}}{h^{5/4}}$$

where $K_1(n, \|H\|_\infty, V(M)) = \max(1, \|H\|_\infty^{\frac{\gamma}{4}} V(M)^{\gamma/4n})$.

Proof. First we have $\left(\int_M \varphi^2 dv\right)^{1/2} \leqslant \|\varphi\|_\infty^{3/4}\left(\int_M \varphi^{1/2} dv\right)^{1/2}$. Moreover

$$\begin{aligned}
\varphi^{1/2} &= |Z|^{1/2}\left|\frac{1}{h^2}(h^2 Z - \delta Z - Hc_\delta \nu) + \frac{1}{h^2}\left(\delta Z + Hc_\delta\nu - h\frac{Z}{|Z|}\right)\right| \\
&\leqslant \frac{|Z|^{1/2}}{nh^2}|X| + \frac{1}{h^2}|Y|
\end{aligned}$$

Then

$$\begin{aligned}
\left(\int_M \varphi^{1/2} dv\right)^{1/2} &\leqslant \frac{1}{n^{1/2}h}\left(\int_M |Z|^{1/2}|X| dv\right)^{1/2} + \frac{1}{h}\left(\int_M |Y| dv\right)^{1/2} \\
&\leqslant \frac{1}{n^{1/2}h}\left(\int_M |Z| dv\right)^{1/4}\|X\|_2^{1/2} + \frac{1}{h}\|Y\|_2^{1/2} V(M)^{1/4} \\
&\leqslant \frac{V(M)^{1/8}}{hn^{1/2}}\|Z\|_2^{1/4}\|X\|_2^{1/2} + \frac{1}{h}\|Y\|_2^{1/2} V(M)^{1/4}
\end{aligned}$$

From the lemmas 3.2, 3.4 and 3.5 we deduce that

$$\begin{aligned}
&\left(\int_M \varphi^{1/2} dv\right)^{1/2} \\
&\leqslant \alpha\left(\frac{\|H\|_\infty^{1/2}}{h^{5/4}} + \frac{1}{h}\left(h + \frac{\|H\|_\infty^2}{h}\max\left(1, \|H\|_\infty^{\gamma} V(M)^{\frac{\gamma}{n}}\right)\right)^{1/4}\right) V(M)^{1/2} C^{1/4} \\
&\leqslant \alpha\left(\frac{\|H\|_\infty^{1/2}}{h^{5/4}} + \frac{1}{h^{3/4}} + \frac{\|H\|_\infty^{1/2}}{h^{5/4}}\max\left(1, \|H\|_\infty^{\frac{\gamma}{4}} V(M)^{\gamma/4n}\right)\right) V(M)^{1/2} C^{1/4} \\
&\leqslant \alpha\frac{\|H\|_\infty^{1/2}}{h^{5/4}}\left(1 + \max(1, \|H\|_\infty^{\frac{\gamma}{4}} V(M)^{\gamma/4n})\right) V(M)^{1/2} C^{1/4}
\end{aligned}$$

where in the last inequality we have used the fact that $h \leqslant \sqrt{2}\|H\|_\infty$. This completes the proof.

$\square$

Lemma 3.8. *Let $\eta > 0$ and*

$$C(n, \|H\|_\infty, \delta, V(M), \eta) = \min\left(\frac{n}{2(n+1)}, \frac{h^2\eta}{\|H\|_\infty^2 K_2^4 K_1^4 V(M)^{4\gamma/n}}\right)$$

where

$$K_2(n, \|H\|_\infty, \delta, V(M), \eta) = \alpha\left(\frac{A_1}{\eta} + \frac{A_2}{\eta^{1/2}} + \|H\|_\infty\right)^\gamma$$

and $A_1 = (\overline{Z}+1)(h+|\delta|^{1/2}\overline{Z})$ and $A_2 = \overline{Z}^{1/2}(h+|\delta|^{1/2}\overline{Z})$. Then if $(P_{C(n,\|H\|_\infty,\delta,V(M),\eta)})$ holds then

$$\|\varphi\|_\infty \leqslant \frac{\eta}{h^3}$$

Proof. We have for any $k \geqslant 2$

$$\frac{1}{2}\int_M \Delta\varphi^2\varphi^{2k-2}dv = \frac{1}{2}\int_M \langle\nabla^M\varphi^2, \nabla^M\varphi^{2k-2}\rangle dv \leqslant 2k\int_M |\nabla^M\varphi|^2\varphi^{2k-2}dv \tag{8}$$

Let us compute $|\nabla^M\varphi|^2$

$$\begin{aligned}
|\nabla^M\varphi|^2 &= \left|\nabla^M\left(|Z|\left(|Z|-\frac{1}{h}\right)^2\right)\right|^2\\
&= \left|\nabla^M|Z|\left(|Z|-\frac{1}{h}\right)^2 + 2|Z|\left(|Z|-\frac{1}{h}\right)\nabla^M|Z|\right|^2\\
&= \left(|Z|-\frac{1}{h}\right)^2\left(3|Z|-\frac{1}{h}\right)^2|\nabla^M|Z||^2\\
&\leqslant 9|Z|^2\left(|Z|-\frac{1}{h}\right)^2|\nabla^M|Z||^2 + \frac{1}{h^2}\left(|Z|-\frac{1}{h}\right)^2|\nabla^M|Z||^2
\end{aligned}$$

A straightforward computation shows that $|\nabla^M|Z||^2 \leqslant c_\delta^2 \leqslant 1+|\delta|\|Z\|_\infty^2$. Then

$$|\nabla^M\varphi|^2 \leqslant A'_1 + A'_2\varphi$$

where $A'_1 = \frac{\alpha}{h^4}(\overline{Z}+1)^2\left(1+\frac{|\delta|}{h^2}\overline{Z}^2\right)$ and $A'_2 = \frac{\alpha}{h}\overline{Z}\left(1+\frac{|\delta|}{h^2}\overline{Z}^2\right)$.
Then reporting this in (8) we get

$$\frac{1}{2}\int_M \Delta\varphi^2\varphi^{2k-2}dv \leqslant 2kA'_1\int_M \varphi^{2k-2}dv + 2kA'_2\int_M \varphi^{2k-1}dv$$

Now, applying the lemma 3.1 we see that if $\|\varphi\|_\infty > \frac{\eta}{h^3}$ then

$$\|\varphi\|_\infty \leqslant \left(\frac{h^3{A'}_1^{1/2}}{\eta} + \frac{h^{3/2}{A'}_2^{1/2}}{\eta^{1/2}} + \|H\|_\infty\right)^\gamma V(M)^{\frac{\gamma}{n}-\frac{1}{2}}\|\varphi\|_2$$

A short computation yields that $h^3{A'}_1^{1/2} \leqslant \alpha(\overline{Z}+1)(h+|\delta|^{1/2}\overline{Z}) = \alpha A_1$ and $h^{3/2}{A'}_2^{1/2} \leqslant \alpha\overline{Z}^{1/2}(h+|\delta|^{1/2}\overline{Z}) = \alpha A_2$. Combining this with the inequality of the lemma 3.7, we deduce that

$$\|\varphi\|_\infty \leqslant \left(\frac{A_1}{\eta} + \frac{A_2}{\eta^{1/2}} + \|H\|_\infty\right)^{4\gamma} V(M)^{4\gamma/n}K_1(n,\|H\|_\infty,V(M))^4\frac{C}{h^5}$$

Now we see that if $C = \min\left(\frac{n}{2(n+1)}, \frac{h^2\eta}{\|H\|_\infty^2 K_2^4K_1^4V(M)^{4\gamma/n}}\right)$ with K_2 as in the lemma then $\|\varphi\|_\infty \leqslant \frac{\eta}{h^3}$.

□

Lemma 3.9. *For any $\varepsilon < \frac{1}{3}$, the pinching condition $(P_{C'_\varepsilon(n,\|H\|_\infty,\delta,V(M))})$ with*

$$C'_\varepsilon(n,\|H\|_\infty,\delta,V(M)) = C(n,\|H\|_\infty,\delta,V(M),\frac{\varepsilon^2}{6})$$

implies

$$\left||Z|-\frac{1}{h}\right|\leqslant\frac{\varepsilon}{h}\quad and\quad\left|r-s_\delta^{-1}\left(\frac{1}{h}\right)\right|\leqslant\frac{\varepsilon}{h}$$

Proof. Consider the function $f(t)=t\left(t-\frac{1}{h}\right)^2$. The function f is increasing on $[0,\frac{1}{3h}]$ and $[\frac{1}{h},+\infty)$ and decreasing on $[\frac{1}{3h},\frac{1}{h}]$. Choose $\eta\leqslant\frac{1}{27}$. From the lemma 3.8 we deduce that the condition $(P_{C(n,\|H\|_\infty,\delta,V(M),\eta)})$ implies that $f(|Z|)\leqslant\frac{\eta}{h^3}\leqslant f\left(\frac{1}{3h}\right)$. Moreover from the lemma 3.1 we see that $\|Z\|_2^2\geqslant\frac{V(M)}{4h^2}$. Then there exists $x_0\in M$ so that $|Z_{x_0}|\geqslant\frac{1}{2h}>\frac{1}{3h}$ and by connexity of M, it follows that $|Z|>\frac{1}{3h}$ over M. Then $\left||Z|-\frac{1}{h}\right|\leqslant\frac{\sqrt{3}\sqrt{\eta}}{h}$. Now

$$\left|r-s_\delta^{-1}\left(\frac{1}{h}\right)\right|\leqslant\left(\sup_I\frac{1}{\sqrt{1-\delta y^2}}\right)\left||Z|-\frac{1}{h}\right|\leqslant\frac{\sqrt{2}\sqrt{3}\sqrt{\eta}}{h}$$

where $I=\mathbb{R}^+$ for $\delta\leqslant 0$ and $I=[0,\frac{1}{\sqrt{2\delta}}]$ for $\delta>0$. Now we conclude by choosing $\eta=\frac{\varepsilon^2}{6}$.

□

We are now in a position to prove the theorem 1.2.

Proof of Theorem 1.2: The case $\delta=0$ is a particular case of [6]. From the lemma above, we know that for any $\varepsilon<\frac{1}{3}$, the pinching (P_{C_ε}) implies that $\phi(M)\subset\overline{B}_p\left(R+\frac{\varepsilon}{h}\right)\setminus B_p\left(R-\frac{\varepsilon}{h}\right)$ with $R=s_\delta^{-1}\left(\frac{1}{h}\right)$. Then $\phi(M)\subset\overline{B}_p(R)\setminus B_p(R-\frac{\varepsilon}{h})$. Let $x_0\in S_p(R)$ so that $\phi(M)\subset\left(\overline{B}_p(R+\frac{\varepsilon}{h})\setminus B_p(R-\frac{\varepsilon}{h})\right)\setminus B_{x_0}(\rho)$ where ρ satisfies

$$t_\delta\left(\frac{R+\rho}{2}\right)-t_\delta\left(\frac{R}{2}\right)=\frac{\varepsilon}{\|H\|_\infty}$$

for $\delta\leqslant 0$ and

$$t_\delta\left(\frac{R}{2}\right)-t_\delta\left(\frac{R-\rho}{2}\right)=\frac{\varepsilon}{\|H\|_\infty}$$

for $\delta>0$, where $t_\delta=\frac{s_\delta}{c_\delta}$. From the proof of the lemma 4.3 of [12] we know that for $\varepsilon<a(n,\|H\|_\infty)$ there exists a point $x_0\in M$ so that $|H(x_0)|\geqslant b(n,\|H\|_\infty)\frac{\|H\|_\infty}{\varepsilon}$. Now if we have chosen $\varepsilon<b(n,\|H\|_\infty)$ we get a contradiction. Then $\phi(M)\cap B_{x_0}(\rho)\neq\emptyset$ and $d_{GH}\left(\phi(M),S\left(p,s_\delta^{-1}\left(\frac{1}{h}\right)\right)\right)<\frac{\varepsilon}{h}$.

□

4. The proof of the diffeomorphism

First we recall that $C'_\varepsilon=\min\left(\frac{n}{2(n+1)},\frac{\alpha\varepsilon^2h^2}{\|H\|_\infty^2(K_2K_1)^4(n,\|H\|_\infty,\delta,V(M),\varepsilon^2)V(M)^{4\gamma/n}}\right)$. Now it is easy to see that

$$\begin{aligned}&C'_\varepsilon(n,\|H\|_\infty,\delta,V(M))\\&\geqslant\min\left(\frac{n}{2(n+1)},\frac{\alpha\varepsilon^2h^2}{\|H\|_\infty^2(K_2K_1)^4(n,(1/\sqrt{n})\|B\|_\infty,\delta,V(M),\varepsilon^2)V(M)^{4\gamma/n}}\right)\end{aligned}$$

Then we consider the new constant

$$\begin{aligned}&C'_\varepsilon(n,\|H\|_\infty,\|B\|_\infty,\delta,V(M))\\&=\min\left(\frac{n}{2(n+1)},\frac{\alpha\varepsilon^2h^2}{\|H\|_\infty^2(K_2K_1)^4(n,(1/\sqrt{n})\|B\|_\infty,\delta,V(M),\varepsilon^2)V(M)^{4\gamma/n}}\right)\end{aligned}$$

and since $\frac{h^2}{\|H\|_\infty^2}\geqslant 1$ for $\delta\geqslant 0$, we obtain a constant which is not depending on $\|H\|_\infty$.

From now we will need a dependence on $\|B\|_\infty$ in order to prove the diffeomorphism and the quasi-isometry.

Now, let us consider $\begin{array}{rccc}F&:&M&\longrightarrow&S\left(p,s_\delta^{-1}\left(\frac{1}{h}\right)\right)\\&&x&\longmapsto&\exp_p\left(s_\delta^{-1}\left(\frac{1}{h}\right)\frac{X}{|X|}\right)\end{array}$, where $X=\exp_p^{-1}(x)$.

For more convenience we will put $\varrho=s_\delta^{-1}\left(\frac{1}{h}\right)\frac{X}{|X|}$.

Lemma 4.1. *Let $u\in T_xM$ so that $|u|=1$ and $v=u-\langle u,\nabla^M r\rangle\nabla^N\ r$. We have*

$$\frac{1}{h^2s_\mu(r)^2}|v|^2\leqslant|dF_x(u)|^2\leqslant\frac{s_\mu\left(s_\delta^{-1}\left(\frac{1}{h}\right)\right)^2}{s_\delta(r)^2}|v|^2$$

Proof. An easy computation shows that

$$d\left(\frac{X}{|X|}\right)|_x(u)\ =\frac{1}{r}d\exp_p^{-1}|_x(u)\ -\ \frac{dr(u)}{r^2}\exp_p^{-1}(x)$$

Then we deduce that

$$\begin{aligned}dF_x(u)&=d\exp_p|_\varrho\left(s_\delta^{-1}\left(\frac{1}{h}\right)d\left(\frac{X}{|X|}\right)|_x(u)\right)\\&=\frac{s_\delta^{-1}\left(\frac{1}{h}\right)}{r}d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(u)\right)\\&-\frac{s_\delta^{-1}\left(\frac{1}{h}\right)dr(u)}{r^2}d\exp_p|_\varrho\left(\exp_p^{-1}(x)\right)\\&=\frac{s_\delta^{-1}\left(\frac{1}{h}\right)}{r}d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(u)\right)-\frac{s_\delta^{-1}\left(\frac{1}{h}\right)dr(u)}{r}\nabla^N\ r\Big|_{F(x)}\end{aligned}$$

Now let us compute the norm of $dF_x(u)$. We have

$$\begin{aligned}|dF_x(u)|^2=\frac{s_\delta^{-1}\left(\frac{1}{h}\right)^2}{r^2}&\Big[\left|d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(u)\right)\right|^2\\&-2\langle d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(u)\right),\nabla^N\ r\rangle_{F(x)}dr(u)+dr(u)^2\Big]\end{aligned}$$

Now since $\exp_p$ is a radial isometry (see for instance [13]), we have

$$\langle d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(u)\right), \nabla^N\ r\rangle_{F(x)} = \langle d\exp_p^{-1}|_x(u)\ , \frac{X}{|X|}\rangle = \langle u, \nabla^N\ r\rangle_x$$

and it follows that

$$|dF_x(u)|^2 = \frac{s_\delta^{-1}\left(\frac{1}{h}\right)^2}{r^2}\left[\left|d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(u)\right)\right|^2 - \langle\nabla^M r, u\rangle^2\right] \tag{9}$$

Now

$$\begin{aligned}\left|d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(u)\right)\right|^2 = \Big|&d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(v)\right)\\ &+\langle u, \nabla^M r\rangle d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(\nabla^N\ r)\right)\Big|^2\end{aligned}$$

where $v = u - \langle u, \nabla^M r\rangle\nabla^N\ r$. Developping this expression we get

$$\begin{aligned}&\left|d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(u)\right)\right|^2 =\\ &\left|d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(v)\right)\right|^2 + \langle u, \nabla^M r\rangle^2\left|d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(\nabla^N\ r)\right)\right|^2\\ &+2\langle u, \nabla^M r\rangle\langle d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(v)\right), d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(\nabla^N\ r)\right)\rangle\\ &= \left|d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(v)\right)\right|^2 + \langle u, \nabla^M r\rangle^2\end{aligned}$$

where in the last equality we have used again the radial isometry property of the exponential map. And reporting this in (9) we obtain

$$|dF_x(u)|^2 = \frac{s_\delta^{-1}\left(\frac{1}{h}\right)^2}{r^2}\left|d\exp_p|_\varrho\left(d\exp_p^{-1}|_x(v)\right)\right|^2$$

Since $\mu \leqslant K^N \leqslant \delta$ the standard Jacobi field estimates (see for instance corollary 2.8, p 153 of [13]) say that for any vector w orthogonal to $\nabla^N\ r$ at y we have

$$|w|^2\frac{r^2}{s_\mu(r)^2} \leqslant |d\exp_p^{-1}|_y(w)\ |^2 \leqslant |w|^2\frac{r^2}{s_\delta(r)^2}$$

This gives

$$\frac{s_\delta(s_\delta^{-1}\left(\frac{1}{h}\right))^2}{r^2}|d\exp_p^{-1}|_x(v)\ |^2 \leqslant |dF_x(u)|^2 \leqslant \frac{s_\mu(s_\delta^{-1}\left(\frac{1}{h}\right))^2}{r^2}|d\exp_p^{-1}|_x(v)\ |^2$$

and applying again the standard Jacobi field estimates we obtain the desired inequalities of the lemma.

□

Lemma 4.2. *Let $u \in T_xM$ so that $|u| = 1$. Then for any $\eta > 0$, there exists a constant $\rho(\delta, \mu, \eta) > 0$ so that if M is contained in the ball $B(p, \rho(\delta, \mu, \eta))$, then*

$$\frac{(1-\eta)^2}{h^2 s_\delta^2(r)}(1 - |\nabla^M r|^2) \leqslant |dF_x(u)|^2 \leqslant \frac{(1+\eta)^2}{h^2 s_\delta^2(r)}$$

Moreover $\rho(\delta, \mu, \eta) \longrightarrow \infty$ when $\delta - \mu \longrightarrow 0$ and $\rho(\delta, \mu, \eta) \longrightarrow 0$ when $\eta \longrightarrow 0$.

Proof. Let $r \geqslant 0$. For $t \in (-\infty, \frac{\pi^2}{16r^2})$, consider the function $\sigma(t) = s_t(r)$. An easy verification yields that σ is C^1 on $(-\infty, \frac{\pi^2}{16r^2})$ and

$$\sigma'(t) = \begin{cases} \frac{r^3 c_t(r)}{2}\left(\frac{\sqrt{t}r - \tan(\sqrt{t}r)}{(\sqrt{t}r)^3}\right) & \text{if } t \in (0, \frac{\pi^2}{16r^2}) \\ -\frac{r^3}{6} & \text{if } \ t = 0 \\ \frac{r^3 c_t(r)}{2}\left(\frac{-\sqrt{-t}r + \tanh(\sqrt{-t}r)}{(\sqrt{-t}r)^3}\right) & \text{if } t \in (-\infty, 0) \end{cases}$$

It follows that σ is decreasing on $(-\infty, \frac{\pi^2}{16r^2})$ and that there exists a constant D so that $|\sigma'(t)| \leqslant D r^3 c_t(r)$, for any $t \in (-\infty, \frac{\pi^2}{16r^2})$. It follows that

$$0 \leqslant s_\mu(r) - s_\delta(r) \leqslant D r^3 c_\mu(r)(\delta - \mu) \tag{10}$$

Now we have

$$\begin{aligned} \frac{1}{h s_\mu(r)} &\geqslant \frac{1}{h\left(s_\delta(r) + D r^3 c_\mu(r)(\delta - \mu)\right)} \\ &\geqslant \frac{1}{h s_\delta(r)(1 + D\left(\frac{r}{s_\delta(r)}\right) r^2 c_\mu(r)(\delta - \mu))} \end{aligned}$$

The function $t \longmapsto \frac{t}{s_\delta(t)}$ beeing bounded on $[0, \infty)$ and on $[0, \frac{\pi}{4\sqrt{\delta}})$ for $\delta > 0$ there exists a constant D' so that

$$\frac{1}{h s_\mu(r)} \geqslant \frac{1}{h s_\delta(r)\left(1 + D' r^2 c_\mu(r)(\delta - \mu)\right)} \tag{11}$$

On the other hand, as we have seen it in the proof of the lemma 3.9, $s_\delta^{-1}\left(\frac{1}{h}\right) \in [0, \frac{\pi}{4\sqrt{\delta}})$ for $\delta > 0$ and we can apply the inequality (10) which gives

$$\begin{aligned} s_\mu\left(s_\delta^{-1}\left(\frac{1}{h}\right)\right) &\leqslant \frac{1}{h} + D\left(s_\delta^{-1}\left(\frac{1}{h}\right)\right)^3 c_\mu\left(s_\delta^{-1}\left(\frac{1}{h}\right)\right)(\delta - \mu) \\ &\leqslant \frac{1}{h}\left[1 + D\left(\frac{s_\delta^{-1}\left(\frac{1}{h}\right)}{1/h}\right) s_\delta^{-1}\left(\frac{1}{h}\right)^2 c_\mu\left(s_\delta^{-1}\left(\frac{1}{h}\right)\right)(\delta - \mu)\right] \end{aligned}$$

And using the same arguments concerning the function $t \longmapsto \frac{t}{s_\delta(t)}$, we have

$$s_\mu\left(s_\delta^{-1}\left(\frac{1}{h}\right)\right) \leqslant \frac{1}{h}\left(1 + D' R^2 \tilde{c}_\mu(R)(\delta - \mu)\right) \tag{12}$$

where

$$\tilde{c}_\mu(r) = \begin{cases} 1 & \text{if } \mu \geqslant 0 \\ c_\mu(r) & \text{if } \ \mu < 0 \end{cases}$$

From the two inequalities (11) and (12) we deduce that there exists a constant $\rho(\delta, \mu, \eta)$ so that if $R \leqslant \rho(\delta, \mu, \eta)$ then $\frac{1}{h s_\mu(r)} \geqslant \frac{1-\eta}{h s_\delta(r)}$ and $s_\mu\left(s_\delta^{-1}\left(\frac{1}{h}\right)\right) \leqslant \frac{1}{h}(1 + \eta)$. Finally from the lemma 4.1 we deduce that

$$\frac{(1-\eta)^2}{h^2 s_\delta^2(r)}|v|^2 \leqslant |dF_x(u)|^2 \leqslant \frac{(1+\eta)^2}{h^2 s_\delta^2(r)}|v|^2$$

Since we have assumed that $|u| = 1$ and $v = u - \langle u, \nabla^M r\rangle \nabla^N r$ we get the desired result.

□

We can now give the proof of the theorem 1.1

Proof of Theorem 1.1: Let $\varepsilon < \frac{1}{3}$. From the lemma 3.9 if $(P_{C'_\varepsilon})$ holds then $\big||Z| - \frac{1}{h}\big| < \frac{\varepsilon}{h}$. From this and the lemma 4.2 we deduce that if M is contained in the ball $B(p, \rho(\delta, \mu, \eta))$ then

$$\left[\frac{(1-\eta)^2}{(1+\varepsilon)^2} - 1\right] - \frac{(1-\eta)^2}{(1+\varepsilon)^2}\|\nabla^M r\|_\infty^2 \leqslant |dF_x(u)|^2 - 1 \leqslant \frac{(1+\eta)^2}{(1-\varepsilon)^2} - 1$$

To complete the proof of the theorem 1.1 we need the following lemma

Lemma 4.3. *Let* $C''_\eta = \frac{\alpha\eta^2}{\left(\left(\frac{1}{\eta}+1\right)\|B\|_\infty + |\mu|^{1/2}\right)^{2\gamma} V(M)^{2\gamma/n}}$ *and* $\tilde{C}_{\varepsilon,\eta} = \min(C'_\varepsilon, C''_\eta)$*. Then for* $\varepsilon < \frac{1}{3}$*,* $(P_{\tilde{C}_{\varepsilon,\eta}})$ *implies that* $\|\nabla^M r\|_\infty \leqslant \eta$*.*

Proof. As usually by computing the Laplacian of $|\nabla^M r|^2$ and using the Bochner formula we get

$$\frac{1}{2}\int_M \Delta|\nabla^M r|^2 |\nabla^M r|^{2k-2} dv \leqslant \int_M (\langle \Delta dr, dr\rangle - \mathrm{Ric}(\nabla^M r, \nabla^M r))|\nabla^M r|^{2k-2} dv$$

Now integrating by part and using the Gauss formula we obtain

$$\begin{aligned}\frac{1}{2}\int_M \Delta|\nabla^M r|^2 |\nabla^M r|^{2k-2} dv \leqslant &\int_M ((\Delta r)^2 - \overline{R}^\phi(\nabla^M r, \nabla^M r) - nHB(\nabla^M r, \nabla^M r) \\ &+ |B\nabla^M r|^2)|\nabla^M r|^{2k-2} dv - \int_M \Delta r \langle \nabla^M r, \nabla^M |\nabla^M r|^{2k-2}\rangle dv\end{aligned}$$

(13)

$$\begin{aligned}\leqslant &\int_M ((\Delta r)^2 |\nabla^M r|^{2k-2} dv + ((n-1)|\mu| + (\sqrt{n}+1)\|B\|_\infty^2)) \int_M |\nabla^M r|^{2k} dv \\ &- 2(k-1)\int_M \Delta r \nabla^M dr(\nabla^M r, \nabla^M r)|\nabla^M r|^{2k-4} dv\end{aligned}$$

Now

$$\begin{aligned}\nabla^M dr(\nabla^M r, \nabla^M r) &= \nabla^N dr(\nabla^M r, \nabla^M r) - B(\nabla^M r, \nabla^M r)\langle \nabla^N r, \nu\rangle \\ &= \nabla^N dr(\nabla^M r - |\nabla^M r|^2 \nabla^N r, \nabla^M r - |\nabla^M r|^2 \nabla^N r) \\ &\quad - B(\nabla^M r, \nabla^M r)\langle \nabla^N r, \nu\rangle\end{aligned}$$

From the comparisons theorems (see for instance [13] p 153) we deduce that

$$|\nabla^M dr(\nabla^M r, \nabla^M r)| \leqslant \left(\frac{c_\mu}{s_\mu} + \|B\|_\infty\right) |\nabla^M r|^2 \tag{14}$$

Similarly, $(e_i)_{1\leqslant i\leqslant n}$ being an orthonormal frame in a neighborhood of the point where we are computing we have

$$\begin{aligned}
|\Delta r| &\leqslant \sum_{i=1}^{n} |\nabla^N\ r(e_i, e_i)| + nH\langle \nabla^N\ r, \nu\rangle \\
&= \sum_{i=1}^{n} |\nabla^N\ (e_i - \langle e_i, \nabla^N\ r\rangle \nabla^N\ r, e_i - \langle e_i, \nabla^N\ r\rangle \nabla^N\ r)| + \sqrt{n}\|B\|_\infty \\
&\leqslant n\frac{c_\mu}{s_\mu} + \sqrt{n}\|B\|_\infty
\end{aligned}$$

and reporting this and (14) in (13) we get

$$\begin{aligned}
\frac{1}{2}\int_M \Delta|\nabla^M r|^2 |\nabla^M r|^{2k-2} dv \leqslant 2k\left(n\frac{c_\mu}{s_\mu} + \sqrt{n}\|B\|_\infty\right)^2 \int_M |\nabla^M r|^{2k-2} dv \\
+ ((n-1)|\mu| + (\sqrt{n}+1)\|B\|_\infty^2) \int_M |\nabla^M r|^{2k} dv
\end{aligned}$$

Now we choose $\rho'(\delta,\mu)$ so that in the ball of radius $\rho'(\delta,\mu)$, $c_\mu(r) \leqslant 1$. Now the pinching condition $(P_{C'_\varepsilon})$ implies that $\frac{c_\mu(r)}{s_\mu(r)} \leqslant \frac{1}{s_\delta(r)} \leqslant \frac{h}{1-\varepsilon}$. And since $\varepsilon < \frac{1}{3}$, $h \leqslant \sqrt{2}\|H\|_\infty$ and $\|H\|_\infty \leqslant \sqrt{n}\|B\|_\infty$ we deduce that $\frac{c_\mu(r)}{s_\mu(r)} \leqslant \|B\|_\infty$ and

$$\begin{aligned}
\frac{1}{2}\int_M \Delta|\nabla^M r|^2 |\nabla^M r|^{2k-2} dv \leqslant \alpha k\|B\|_\infty^2 \int_M |\nabla^M r|^{2k-2} dv \\
+ \alpha(|\mu| + \|B\|_\infty^2) \int_M |\nabla^M r|^{2k} dv
\end{aligned}$$

where α is a constant depending only on n. Then from the proposition 3.1, if $\|\nabla^M r\|_\infty > \eta$ then

$$\|\nabla^M r\|_\infty \leqslant \alpha\left(\left(1+\frac{1}{\eta}\right)\|B\|_\infty + |\mu|^{1/2}\right)^\gamma V(M)^{\frac{\gamma}{n}-\frac{1}{2}} \|\nabla^M r\|_2$$

Now let $\varepsilon < 1/3$ and assume that we have $(P_{C'_\varepsilon})$. Then note that $|\nabla^M r|^2 \leqslant \frac{1}{s_\delta^2(r)}|Z^T|^2 \leqslant \frac{h^2}{(1-\varepsilon)^2}|Z^T|^2 \leqslant \frac{9}{2}\|H\|_\infty |Z^T|^2$ and from the lemmas 3.1 and 3.2 $\|\nabla^M r\|_2 \leqslant 3V(M)^{1/2} C_\varepsilon'^{1/2}$. Therefore

$$\|\nabla^M r\|_\infty \leqslant \alpha\left(\left(1+\frac{1}{\eta}\right)\|B\|_\infty + |\mu|^{1/2}\right)^\gamma V(M)^{\frac{\gamma}{n}} C_\varepsilon'^{1/2}$$

□

Now we are allowed to complete the proof of theorem 1.1. First let us choose $\eta = \varepsilon$ and put $C_\varepsilon = \tilde{C}_{\varepsilon,\varepsilon}$. Then if $\phi(M) \subset B(p, R(\delta,\mu,\varepsilon))$ (with $R(\delta,\mu,\varepsilon) = \min(\rho(\delta,\mu,\varepsilon), \rho'(\delta,\mu)))$ and if (P_{C_ε}) is satisfied then

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 - 1 - \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 \varepsilon^2 \leqslant |dF_x(u)|^2 - 1 \leqslant \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 - 1$$

since $\varepsilon < \frac{1}{3}$ it is easy to see that $||dF_x(u)|^2 - 1| \leqslant 6\varepsilon$. From the choice of ε, we deduce that F is a diffeomorphism and from the definition of the Gromov-Hausdorff distance it follows that we have also

$$d_{GH}\left(\phi(M), S\left(p, s_\delta^{-1}\left(\frac{1}{h}\right)\right)\right) < \frac{\varepsilon}{h}$$

Then choosing $\varepsilon < 1/18$ we obtain the desired result.

Now to complete the proof, from the expression of C_ε the assertion (1) and (2) are obvious. Now to prove that if $V(M)^{1/n}\|B\|_\infty \leqslant v$, then

$$h^2 C_\varepsilon(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta) \longrightarrow \infty$$

when $\|H\|_\infty \longrightarrow \infty$ it is sufficient to notice that $K_1 \leqslant \max(1, v^{1/4})$, $K_2 V(M)^{\gamma/n} \leqslant \alpha\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + 1\right)^\gamma v^\gamma$ and $C'_\varepsilon \geqslant \frac{\alpha\varepsilon^2}{\left(\frac{1}{\varepsilon}+1+\left|\frac{\mu}{\delta}\right|^{1/2}\right)^{2\gamma} v^{2\gamma}}$.

□

5. Application to the stability

Briefly, we recall the problem of the stability of hypersurfaces with constant mean curvature (see for instance [3]).

Let (M^n, g) be an oriented compact n -dimensional hypersurface isometrically immersed by ϕ in a $n+1$-dimensional oriented manifold (N^{n+1}, h). We assume that M is oriented by the global unit normal field ν so that ν is compatible with the orientations of M and N. Let $F : (-\varepsilon, \varepsilon) \times M \longrightarrow N$ be a variation of ϕ so that $F(0,.) = \phi$. We recall that the balance volume is the function $V : (-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}$ defined by

$$\int_{[0,t]\times M} F^\star dv_h$$

where dv_h is the element volume associated to the metric h. It is well known that

$$V'(0) = \int_M f dv$$

where $f(x) = \langle \frac{\partial F}{\partial t}(0,x), \nu\rangle$. Moreover the area function $A(t) = \displaystyle\int_M dv_{F_t^\star h}$ satisfies

$$A'(0) = -n\int_M Hf dv$$

The balance volume V is said to be preserving volume if $V(t) = V(0)$ in a neighborhood of 0 ; in this case we have $\displaystyle\int_M f dv = 0$. Conversely, for all smooth

function f so that $\int_M f dv = 0$, there exists a preserving volume variation so that $f = \langle \frac{\partial F}{\partial t}(0,x), \nu\rangle$. The following assertions are equivalent

(1) The immersion ϕ is a critical point of the area (i.e. $A'(0) = 0$) for all variation with preserving volume.
(2) $\int_M H f dv = 0$ for any smooth function so that $\int_M f dv = 0$.
(3) There exists a constant H_0 so that $A'(0) + nH_0 V'(0) = 0$ for any variation.
(4) ϕ is of constant mean curvature H_0.

An immersion with constant mean curvature H_0 will be said stable if $A''(0) \geqslant 0$ for all preserving volume variation. Now we consider the function $J(t)$ defined by

$$J(t) = A(t) + nH_0 V(t)$$

Then $J''(0)$ is depending only on f and we have

$$J''(0) = \int_M |df|^2 dv - \int_M (Ric^N(\nu,\nu) + |B|^2) f^2 dv$$

where Ric^N is the Ricci curvature of N with respect to the metric h. It is known that ϕ is a stable constant mean curvature immersion if and only if $J''(0) \geqslant 0$ for any smooth function so that $\int_M f dv = 0$.

Now let us give a proof of the theorem 1.3.

Proof of Theorem 1.3: Let f be the first eigenfunction associated to $\lambda_1(M)$. Since $\int_M f dv = 0$ then $J''(0) \geqslant 0$ and

$$\lambda_1(M) \int_M f^2 dv - \int_M (Ric^N(\nu,\nu) + nH^2 + |\tau|^2) f^2 dv \geqslant 0$$

where τ is the umbilicity tensor (i.e. $\tau = nHg - B$). Since $\mu \leqslant K^N \leqslant \delta$, we deduce that

$$n(H^2 + \mu) \leqslant \lambda_1(M) \leqslant n(H^2 + \delta)$$

In other words, we have the pinching condition

$$n(H^2 + \delta) - n(\delta - \mu) \leqslant \lambda_1(M) \leqslant n(H^2 + \delta)$$

Now fix $\varepsilon < \frac{1}{18}$ and let $R > 0$ so that $\phi(M)$ lies on a ball or radius R. Let ρ be the extrinsic radius of M (i.e. the radius of the smallest ball containing $\phi(M)$). Then $t_\delta(R) \geqslant t_\delta(\rho)$. On the other hand, we know that $t_\delta(\rho) \geqslant \frac{1}{\|H\|_\infty} \geqslant \frac{1}{\|B\|_\infty}$ (see [2]). If we assume that $V(M)^{1/n} \leqslant \frac{v}{\|B\|_\infty}$, we see that $M \in \mathcal{H}(n,N)$ for R small enough. On the other hand from the theorem 1.1 it follows that $h^2 C_\varepsilon(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta) \longrightarrow \infty$ when $R \longrightarrow 0$ and there exists $R'_\varepsilon(\delta,\mu,v,i(N))$ so that if $\phi(M)$ lies in a ball of radius $R'_\varepsilon(\delta,\mu,v,i(N)))$ then $C_\varepsilon(n, \|H\|_\infty, \|B\|_\infty, V(M), \delta) \geqslant (\delta - \mu)/h^2$. Now we conclude by putting $R_\varepsilon(\delta,\mu,v,i(N)) = \min(\frac{1}{2}R(\delta,\mu,\varepsilon), R'_\varepsilon(\delta,\mu,v,i(N)))$ (the quantity $R(\delta,\mu,\varepsilon)$ is defined in the theorem 1.1). Then $\phi(M)$ is contained in the ball $B(p, R(\delta,\mu,\varepsilon))$ where p is the center of mass of M and the conclusions of the theorem 1.1 are valid.

□

6. Application to the almost umbilic hypersurfaces

The theorems 1.4 and 1.5 are obtained by combining the theorem 1.1 and the eigenvalue pinching theorems of [6] with an eigenvalue pinching result in almost positive Ricci curvature due to E. Aubry ([1]).

In the following theorem we denote $\underline{\mathrm{Ric}}(x)$ the lowest eigenvalue of the Ricci tensor $\mathrm{Ric}(x)$ at $x \in M$. Moreover for any function f, we put $f_- = \min(-f, 0)$.

Theorem 6.1. *(*E.Aubry*) Let (M^n, g) be a complete n-dimensional Riemannian manifold and $q > n/2$. If M has finite volume and*

$$\rho_q = \frac{1}{kV(M)^{1/q}} \left(\int_M (\underline{Ric} - (n-1)k)_-^q \, dv \right)^{1/q} \leqslant C(q,n)^{-1/q}$$

then M is compact and $\lambda_1(M) \geqslant nk(1 - C(q,n)\rho_q)$.

Proof of Theorems 1.4 and 1.5: Using Gauss formula and the fact that N is of constant sectional curvature δ, we have

$$\begin{aligned}
\frac{\|\mathrm{Ric} - (n-1)(H^2+\delta)g\|_q}{V(M)^{1/q}} &= \frac{\|\overline{R}^\phi + nHB - B^2 - (n-1)H^2 g - (n-1)\delta g\|_q}{V(M)^{1/q}} \\
&= \frac{\|(n-2)H\tau - \tau^2\|_q}{V(M)^{1/q}} \\
&\leqslant \frac{(n-2)\|H\|_\infty \|\tau\|_{2q}}{V(M)^{1/2q}} + \frac{\|\tau\|_{2q}^2}{V(M)^{1/q}}
\end{aligned}$$

Now, putting $k = \frac{\|H\|_{2r}^2}{V(M)^{1/r}} + \delta$ we get

$$\begin{aligned}
\frac{\|\mathrm{Ric} - (n-1)kg\|_q}{V(M)^{1/q}} &\leqslant \frac{\|\mathrm{Ric} - (n-1)(H^2+\delta)g\|_q}{V(M)^{1/q}} + \frac{(n-1)\sqrt{n}}{V(M)^{1/q}} \left\| H^2 - \frac{\|H\|_{2r}^2}{V(M)^{1/r}} \right\|_q \\
&\leqslant \frac{(n-2)\|H\|_\infty \|\tau\|_{2q}}{V(M)^{1/2q}} + \frac{\|\tau\|_{2q}^2}{V(M)^{1/q}} + \frac{(n-1)\sqrt{n}}{V(M)^{1/q}} \left\| H^2 - \frac{\|H\|_{2r}^2}{V(M)^{1/r}} \right\|_q
\end{aligned}$$

If $\|\tau\|_{2q} \leqslant \eta_{1,\varepsilon}$ and $\left\| H^2 - \frac{\|H\|_{2r}^2}{V(M)^{1/r}} \right\|_q \leqslant \eta_{2,\varepsilon}$ then

$$\frac{\|\mathrm{Ric} - (n-1)kg\|_q}{V(M)^{1/q}} \leqslant \alpha \|H\|_\infty^2 (\eta_{1,\varepsilon} + \eta_{2,\varepsilon})$$

and if $\eta_{i,\varepsilon} \leqslant \frac{k}{2\alpha\|H\|_\infty^2} \min\left(C(q,n)^{-1/q}, \frac{C_\varepsilon(n, \|H\|_\infty, \|B\|_\infty, V(M))}{C(n,q)} \right)$. The theorem 6.1 allows us to conclude that

$$\lambda_1(M) \geqslant n \left(\frac{\|H\|_{2r}^2}{V(M)^{1/r}} + \delta \right) (1 - C_\varepsilon)$$

Now the conclusion is immediate from the pinching theorems of this paper and [6].

□

References

[1] Aubry E. Diameter pinching in almost positive Ricci curvature, *Preprint.*

[2] Baikoussis C., Koufogiorgos T. The diameter of an immersed Riemannian manifold with bounded mean curvature, *J. Austral. Soc.*, (Series A), **31**, (1981), 189-192.

[3] Barbosa J.L., do Carmo M. Stability of Hypersurfaces with Constant Mean Curvature, *Math. Z.*, **185**, No. 3, (1984), 339-353.

[4] Burago D., Zalgaller V. A. Geometric Inequalities, *Springer-Verlag.*

[5] Chavel I.: Riemannian geometry-A modern introduction, *Cambridge university press.*

[6] Colbois B.,Grosjean J.F. A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space, *Comment. Math. Helv.*, **82**, (2007), 175-195.

[7] El Soufi A., Ilias S., Une inégalité du type "Reilly" pour les sous-variétés de l'espace hyperbolique, *Comm. Math. Helv.*, **67**, (1992), 167-181.

[8] Heintze E.: Extrinsic upper bound for λ_1, *Math.Ann.*, **280**, (1988), 389-402.

[9] Hoffman D., Spruck J., Sobolev and isoperimetric inequalities for Riemannian submanifolds, *Comm. Pure and Appl. Math.*, **27**, (1974), 715-727.

[10] Hoffman D., Spruck J., Sobolev and isoperimetric inequalities for Riemannian submanifolds, Erratum *Comm. Pure and Appl. Math.*, **28**, (1975), 765-766.

[11] Reilly R., On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space, *Comment. Math. Helv.*, **52**, (1977), 525-533.

[12] Roth J., Extrinsic radius pinching for hypersurfaces of space forms, to appear in *Differ. Geom. Appl.*

[13] Sakai T.: Riemannian geometry, *A.M.S. translations of Math. Monographs* , **149** , (1996).

[14] Shiohama K., Xu H., Rigidity and sphere theorems for submanifolds, *Kyushu J. Math.*, **48**, (1994), no 2, 291-306.

[15] Shiohama K., Xu H., Rigidity and sphere theorems for submanifolds II, *Kyushu J. Math.*, **54**, (2000), no 1, 103-109.

[16] Xu H.W., Some results on geometry of Riemannian submanifolds, *Ph. D. dissertation, Fudan University*, (1990).

(J.-F. Grosjean) Institut Élie Cartan (Mathématiques), Université Henri Poincaré Nancy I, B.P. 239, F-54506 Vandœuvre-les-Nancy cedex, France

E-mail address: `grosjean@iecn.u-nancy.fr`